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PARTIAL DUALITY IN SU(N) YANG-MILLS THEORY

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Recently we have proposed a set of variables for describing the infrared limit of four dimensional $SU(2)$ Yang-Mills theory. Here we extend these variables to the general case of four dimensional $SU(N)$ Yang-Mills theory. We find that the $SU(N)$ connection A_μ decomposes according to irreducible representations of $SO(N-1)$, and the curvature two form $F_{\mu\nu}$ is related to the symplectic Kirillov two forms that characterize irreducible representations of $SU(N)$. We propose a general class of nonlinear chiral models that may describe stable, soliton-like configurations with nontrivial topological numbers.

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Recently [1] we have proposed a novel decomposition of the four dimensional $SU(2)$ Yang-Mills connection A^a_μ. In addition of a three component unit vector n^a, it involves an abelian gauge field C_μ and a complex scalar $\phi = \rho + i\sigma$. The fields C_μ and ϕ determine a $U(1)$ multiplet under $SU(2)$ gauge transformations in the direction n^a,

$$A^a_\mu \;=\; C_\mu n^a \;+\; \epsilon_{abc}\partial_\mu n^b n^c \;+\; \rho\partial_\mu n^a \;+\; \sigma\epsilon^{abc}\partial_\mu n^b n^c \tag{1}$$

In four dimensions this decomposition is complete in the sense that it reproduces the Yang-Mills equations of motion [1]. This is already suggested by the number of independent fields: if we account for the $U(1)$ invariance, (1) describes $D+2$ field degrees of freedom. For $D=4$ this equals $3(D-2)$, the number of transverse polarization degrees of freedom described by a $SU(2)$ connection in D dimensions. Furthermore, if we properly specify the component fields, (1) reduces to several known $SU(2)$ field configurations. As an example, if we set $A_0 = C_\mu = \rho = \sigma = 0$ and specify n^a to coincide with the (singular) radial vector x^a/r, the parametrization (1) yields the (singular) Wu-Yang monopole configuration [2]. As another example, if we specify a rotation symmetric configuration with $C_i = Cx^i$, and C_0, C, ϕ to depend on r and t only, and set $n^a = x^a/r$, (1) reduces to Witten's Ansatz for instantons [3].

Here we wish to generalize (1) to $SU(N)$ We shall argue that exactly in four dimensions the $SU(N)$ Yang-Mills connection admits the following decomposition

$$A^a_\mu \;=\; C^i_\mu m^a_i \;+\; f^{abc}\partial_\mu m^b_i m^c_i \;+\; \rho^{ij}\, f^{abc}\partial_\mu m^b_i m^c_j \;+\; \sigma^{ij}\, d^{abc}\partial_\mu m^b_i m^c_j \tag{2}$$

With $i = 1,...,N-1$ we label the Cartan subalgebra. We shall construct the $N-1$ mutually orthogonal unit vector fields m^a_i (with $a = 1,...,N^2-1 = Dim[SU(n)]$ in the following) so that they describe $N(N-1)$ independent variables. The combination

$$A^a_\mu \;=\; C^i_\mu m^a_i \;+\; f^{abc}\partial_\mu m^b_i m^c_i \tag{3}$$

is the $SU(N)$ Cho connection [4], under the $N-1$ independent gauge transformations generated by the Lie-algebra elements

$$\alpha_i \;=\; \alpha_i m^a_i T^a \tag{4}$$

the vector fields C^i_μ transform as U(1) connections

$$C^i_\mu \;\rightarrow\; C^i_\mu + \partial_\mu \alpha_i$$

Consequently these fields describe $(D-2)(N^2-1)$ independent variables. The $\phi_{ij} = \rho_{ij} + i\sigma_{ij}$ are $N(N-1)$ independent complex scalars, mapped onto each other by the $U(1)$ gauge transformations (4). As a consequence (C^i_μ, ϕ_{ij}) can be viewed as a collection of abelian Higgs multiplets. We shall find that the fields ρ_{ij} and σ_{ij} decompose according to the traceless symmetric tensor, the antisymmetric tensor, the vector and the singlet representations of $SO(N-1)$. In D dimensions (2) then describes

$$N(N-1) \; + \; (D-2)(N-1) \; + \; N(N-1) \; = \; 2N^2 \; + \; (D-4)N \; + \; (2-D)$$

independent variables. In *exactly* $D = 4$ this coincides with $(D-2)(N^2-1)$ which is the number of independent transverse variables described by a $SU(N)$ Yang-Mills connection A_μ^a.

We also observe that for $D = 3$, the number of independent variables in (2) coincides with the dimension of the $SU(N)$ gauge orbit, *independently* of the Yang-Mills equations of motion.

We note that a decomposition of the $SU(N)$ connection has been recently considered by Periwal [5]. It appears that his results are different from ours.

We proceed to the justification of the decomposition (2). For this we need a number of group theoretical relations. Some of these relations seem to be novel, suggesting that further investigations of (2) could reveal hitherto unknown structures.

The defining representation of the $SU(N)$ Lie algebra consists of N^2-1 traceless hermitean $N \times N$ matrices T^a with

$$T^a T^b \; = \; \frac{1}{2N}\delta^{ab} \; + \; \frac{i}{2} f^{abc} T^c + \frac{1}{2} d^{abc} T^c$$

normalized to

$$(T^a, T^b) \; \equiv \; Tr(T^a T^b) \; = \; \frac{1}{2}\delta^{ab}$$

The f^{abc} are (completely antisymmetric and real) structure constants. The d^{abc} are the (completely symmetric) coefficients

$$\frac{1}{2} d^{abc} \; = \; Tr(T^a\{T^b, T^c\}) \; \equiv \; Tr(T^a(T^b T^c + T^c T^b))$$

and we note that in the defining representation we can select the Cartan subalgebra so that

$$\sum_{i,j=1}^{N-1} d^{ijk} d^{ijl} \; = \; \frac{2(N-2)}{N}\delta_{kl} \tag{5}$$

For any four matrices $A, \; B, \; C, \; D$ we have

$$Tr(\; [A,B]\{C,D\} + [A,C]\{B,D\} + [A,D]\{B,C\} \;) \; = \; 0$$

and

$$Tr(\; [A,B][C,D] + \{A,C\}\{B,D\} - \{A,D\}\{B,C\} \;) \; = \; 0$$

Hence, if we introduce the matrices $(\mathcal{F}^a)^{bc} = f^{abc}$ (which define the adjoint representation) and $(\mathcal{D}^a)^{bc} = d^{abc}$, we have

$$[\mathcal{F}^a, \mathcal{D}^b] \; = \; -f^{abc}\mathcal{D}^c \tag{6}$$

and

$$[\mathcal{D}^a, \mathcal{D}^b]_{cd} \;=\; f^{abe}\mathcal{F}^e_{cd} \;+\; \frac{2}{N}(\delta^{ad}\delta^{bc} \;-\; \delta^{ac}\delta^{bd}) \tag{7}$$

Note that the $\mathcal{D}^a$ are traceless.

We conjugate the Cartan matrices H_i of the defining representation by a generic element $g \in SU(N)$. This produces set of Lie algebra valued vectors

$$m_i \;=\; m_i^a T^a \;=\; g H_i g^{-1} \tag{8}$$

an over-determined set of coordinates on the orbit $SU(N)/U(1)^{N-1}$: By construction the m_i^a depend on $N(N-1)$ independent variables, they are orthonormal,

$$(m_i, m_j) \;=\; m_i^a m_j^a \;=\; \delta_{ij}$$

and it is straightforward to verify that

$$[m_i, m_j] \;=\; 0 \tag{9}$$

$$\{m_i, m_j\} \;=\; d^{ijk} m_k \tag{10}$$

$$Tr(m_i \partial_\mu m_j) \;=\; (m_i, \partial_\mu m_j) \;=\; 0 \tag{11}$$

We can also show that

$$-\, f^{acd} m_i^c f^{deb} m_i^e \;=\; \delta^{ab} - m_i^a m_i^b \tag{12}$$

is a projection operator. This can be verified *e.g.* by using explicitly the defining representation of $SU(N)$. The result follows since the weights of $SU(N)$ have the same length and the angles between different weights are equal. Notice that (12) can also be represented covariantly, using the $SU(N)$ permutation operator

$$-\,[m_i, T^d] \otimes [m_i, T^d] \;=\; T^b \otimes T^a \;-\; m_i \otimes m_i \tag{13}$$

We now consider the Maurer-Cartan one-form

$$dgg^{-1} \;=\; \omega_{a\mu} T^a dx^\mu$$

We find

$$dm_i \;=\; [\omega, m_i] \tag{14}$$

We now recall that each Cartan matrix H_i can be used to construct a symplectic (Kirillov) two-form on the orbit $SU(N)/U(1)^{N-1}$,

$$\Omega^i \;=\; Tr(\; H_i[g^{-1}dg, g^{-1}dg]\;) \tag{15}$$

$$d\Omega^i \;=\; 0$$

The Ω^i are related to the representations of $SU(N)$; According to the Borel-Weil theorem each linear combination of Ω_i

$$\sum_i n_i \Omega^i \tag{16}$$

with integer coefficients corresponds to an irreducible representation of $SU(N)$.

When we combine the projection operator (12) with the relation (14) and the Jacobi identity, we find that these two-forms Ω^i can be represented in terms of the m_i,

$$\Omega^i \ = \ Tr(m_i[dm_k, dm_k]) \ = \ (m_i, dm_k \wedge dm_k) \tag{17}$$

Explicitly, in components

$$\Omega_{i,\mu\nu} \ = \ f^{abc} m_i^a \partial_\mu m_k^b \partial_\nu m_k^c \tag{18}$$

Next, we proceed to consider a generic Lie algebra element $v = v^a T^a$. We define the (infinitesimal) adjoint action δ^i of the m_i on v by

$$\delta^i v \ = \ [v, m_i] \tag{19}$$

Using (12) and by summing over i we find that this yields (up to a sign) a projection operator to a subspace which is orthogonal to the maximal torus and is spanned by the m_i,

$$(\delta^i)^2 v \ = - \ v + m_i(m_i, v) \tag{20}$$

We shall also need a (local) basis of Lie-algebra valued one-forms in the subspace to which (20) projects. For this we first use (10) to conclude that for the symmetric combination

$$\{dm_i, m_j\} \ + \ \{dm_j, m_i\} \ = \ d_{ijk} dm_k \tag{21}$$

and using (5) we invert this into

$$dm_k \ = \ \frac{N}{2(N-2)} d_{kij}(\ \{dm_i, m_j\} \ + \ \{dm_j, m_i\} \ )$$

Consequently the symmetric combination (21) yields the $SO(N-1)$ vector one-form

$$X^i \ = \ X^i_\mu dx^\mu \ = \ \partial_\mu m_i^a T^a dx^\mu \ = \ \frac{N}{2(N-2)} d_{ijk}(\ \{dm_j, m_k\} + \{dm_k, m_j\} \ ) \tag{22}$$

The antisymmetric combination yields a $SO(N-1)$ antisymmetric tensor one-form

$$Y^{ij} \ = \ Y^{ij}_\mu dx^\mu \ = -Y^{ji}_\mu dx^\mu \ = \ \{dm_i, m_j\} \ - \ \{dm_j, m_i\} \tag{23}$$

$$= \ d^{abc}(\partial_\mu m_i^a m_j^b - \partial_\mu m_j^a m_i^b) \ T^c dx^\mu \tag{24}$$

Finally, we define the $SO(N-1)$ symmetric tensor one-form

$$Z^{ij} \ = \ Z^{ij}_\mu dx^\mu \ = \ Z^{ji}_\mu dx^\mu \ = \ [dm_i, m_j] \ = \ f^{abc} \partial_\mu m_i^a m_j^b \ T^c dx^\mu \tag{25}$$

Under $SO(N-1)$ this decomposes into the traceless symmetric tensor representation and the trace *i.e.* singlet representation, but we use it as is.

Observe that that (22), (24) and (25) are the *only* invariant one-forms that can be constructed using the variables m_i and natural $SU(N)$ invariant concepts. In particular, X^i, Y^{ij} and Z^{ij} are orthogonal to m^i, hence they determine a basis in the corresponding subspace $SU(N)/U(1)^{N-1}$. We can identify them as a basis of roots in $SU(N)$.

The dimension of the $SO(N-1)$ vector representation (22) is $N-1$. The dimension of the antisymmetric tensor representation (24) is $\frac{1}{2}(N-1)(N-2)$. The sum of these coincides with the dimension of (25),

$$N-1 \ + \ \frac{1}{2}(N-1)(N-2) \ = \ \frac{1}{2}N(N-1)$$

Moreover, we find that the $U(1)$ generators (19) map X^i_μ and Y^{ij}_μ into Z^{ij}_μ and *vice versa*,

$$\delta^i X^j \ = \ Z^{ij} \tag{26}$$

$$\delta^i Y^{jk} \ = \ d^{ikl} Z^{jl} \ - \ d^{ijl} Z^{kl} \tag{27}$$

and

$$\begin{aligned} \delta^i Z^{jk} \ &= \ -\frac{1}{N}(\delta_{ij} X^k + \delta_{ik} X^j) \\ + \ \frac{1}{4}(d_{jkl} d_{lin} - d_{jil} d_{lkn} - d_{kil} d_{ljn}) X^n \ &+ \ \frac{1}{4}(d_{jkl} Y^{il} + d_{jil} Y^{kl} + d_{kil} Y^{jl}) \end{aligned} \tag{28}$$

We note that this determines a natural complex structure.

We have now constructed four different sets of $SU(N)$ Lie-algebra valued forms (in μ) from the m_i. Each of these four sets induces an irreducible representation of $SO(N-1)$, they decompose into the vector, the antisymmetric tensor, and the traceless symmetric tensor plus scalar representations of $SO(N-1)$. The cotangent bundle to the co-adjoint orbit $SU(N)/U(1)^{N-1}$ is spanned by the one-forms (in μ) X^i, Y^{ij} and Z^{ij}.

By construction $(m_i, X^i, Y^{ij}, Z^{ij})$ yields a complete set of basis states for the $SU(N)$ Lie algebra, and can be used to decompose generic $SU(N)$ connections. For this we need appropriate dual variables that appear as coefficients. We first note that the connection A^a_μ is a $SU(N)$ Lie-algebra valued one-form, and the $SO(N-1)$ acts on it trivially. Consequently the variable which is dual to m_i must be a one-form which transforms as a vector under $SO(N-1)$. We call it C^i. The variables which are dual to the (X^i, Y^{ij}, Z^{ij}) are zero-forms, and in order to form invariant combinations they must decompose under the action of $SO(N-1)$ in the same manner as X^i, Y^{ij} and Z^{ij}. Since we have also found a natural complex structure which is determined by the δ^i, this suggests that we denote these dual variables by $\phi^{ij} = \rho^{ij} + i\sigma^{ij}$. Here ρ^{ij} is dual to the Z^{ij} and can be decomposed into a traceless symmetric tensor and a singlet under $SO(N-1)$. The σ^{ij} is dual to the X^i and Y^{ij}. It decomposes into a vector and an antisymmetric tensor under $SO(N-1)$. The ($SO(N-1)$ invariant) $SU(N)$ connection A^a_μ then decomposes into

$$A \ = \ C \cdot m \ + \ (1+\rho)[dm, m] \ + \ \sigma\{dm, m\} \tag{29}$$

Exactly in four dimensions this contains the correct number of independent variables for a general $SU(N)$ connection.

The ensuing curvature two form $F = dA + AA$ is obtained by a direct substitution. Of particular interest is the structure of F in the direction m_i, the maximal torus,

$$F^a_{\mu\nu} \ = \ m^a_i(\partial_\mu C^i_\nu - \partial_\nu C^i_\mu) \ - \ m^a_i \Omega^i_{\mu\nu} \ + \ ... \tag{30}$$

Here the Ω^i are the Kirillov two forms (17), the terms that we have not presented explicitely depend on ϕ_{ij} and/or are in the direction of $SU(N)/U(1)^{N-1}$. (If we evaluate the curvature two form for the $SU(N)$ Cho connection (3), we find exactly the terms in (30).) Consequently F is a generating functional for the Kirillov two forms. In particular, for a flat connection we have

$$dC^i = \Omega^i$$

In [1], using Wilsonian renormalization group arguments we suggested that for $SU(2)$ (1) the following action

$$S \ = \ \int d^4x \ (\partial_\mu n)^2 \ + \ \frac{1}{e^2}(n, dn \wedge dn)^2 \tag{31}$$

may be relevant in the infrared limit of Yang-Mills theory. This is interesting since (31) describes knotlike configurations as solitons. The (self)linking of these knots is computed by the Hopf invariant

$$Q \ = \ \int F \wedge A \tag{32}$$

where

$$F \ = \ dA \ = \ (n, dn \wedge dn)$$

The present construction suggests a natural generalization of (31) to $SU(N)$,

$$S \ = \ \int d^4x \ (\partial_\mu m_i)^2 \ + \ \frac{1}{e_i^2}(\ [\partial_\mu m_i, \partial_\nu m_i] \ )^2 \tag{33}$$

which reduces to (31) for $N = 2$. Since $\pi_3(SU(N)/U(1)^{N-1}) = Z$ we expect that in the general case of $SU(N)$ we also have solitons. It would be interesting to understand their detailed structure.

As an example we consider $SU(3)$, the gauge group of strong interactions. There are two $SU(3)$ Lie-algebra valued vectors m_i^a which we denote by m^a and n^a respectively. We have $m^2 = n^2 = 1$ and $m^a n^a = 0$, and (2) becomes

$$\begin{aligned} A_\mu^a &= B_\mu m^a + C_\mu n^a + f^{abc}\partial_\mu m^b m^c + f^{abc}\partial_\mu n^b n^c && (34) \\ &+ \rho_{mm} f^{abc}\partial_\mu m^b m^c + \rho_{mn} f^{abc}\partial_\mu m^b n^c + \rho_{nn} f^{abc}\partial_\mu n^b n^c && (35) \\ &+ \sigma_{mm}\partial_\mu m^a + \sigma_{nn}\partial_\mu n^a + \sigma_{mn} d^{abc}\partial_\mu m^b n^c && (36) \end{aligned}$$

Here (34) is the SU(3) Cho connection, and (35), (36) are the components of A_μ^a in the direction of the orbit $SU(3)/U(1) \times U(1)$. These components transform into each other under the action of the operators

$$\delta_m = [\,\bullet\,, m\,] \qquad \delta_n = [\,\bullet\,,\, n\,]$$

In the Gell-Mann basis the vectors m and n satisfy

$$\begin{aligned} [m,n] &= 0 & \{m,m\} &= \frac{1}{\sqrt{3}}n \\ \{n,n\} &= -\frac{1}{\sqrt{3}}n & \{m,n\} &= \frac{1}{\sqrt{3}}m \end{aligned}$$

Notice that n is represented in terms of m, a unit vector with six independent field degrees of freedom to parametrize $SU(3)/U(1) \times U(1)$.

The action of δ_m and δ_n on $SU(3)/U(1) \times U(1)$ can be diagonalized. We find that δ_m corresponds to the adjoint action of the Cartan element λ_3 and δ_n to the adjoint action of the Cartan element λ_8. This decomposes the $SU(3)/U(1) \times U(1)$ components of A_μ^a into $U(1)$ multiplets. In particular, $\delta_m^2 + \delta_n^2$ is a projection operator onto the basis one-forms of $SU(3)/U(1) \times U(1)$. Finally, the Kirillov symplectic two-forms are

$$\Omega_m = f^{abc} m^a (\partial_\mu m^b \partial_\nu m^c + \partial_\mu n^b \partial_\nu n^c)$$

and

$$\Omega_n = f^{abc} n^a (\partial_\mu m^b \partial_\nu m^c + \partial_\mu n^b \partial_\nu n^c)$$

and the action (33) is

$$S = \int d^4x \left((\partial_\mu m)^2 + (\partial_\mu n)^2 + \frac{1}{e_m^2}(f^{abc}\partial_\mu m^b \partial_\nu m^c)^2 + \frac{1}{e_n^2}(f^{abc}\partial_\mu n^b \partial_\nu n^c)^2 \right)$$

In conclusion, we have presented a group theoretical decomposition of four dimensional $SU(N)$ connection A_μ^a, which we argue is complete in the sense described in [1]. This decomposition involves a number of natural group theoretical concepts, and in particular relates the $SU(N)$ curvature two form to the Kirillov symplectic two forms on the $SU(N)$ coadjoint orbits. Curiously, we also find that the components of A in the direction of these orbits can be decomposed according to irreducible representations of $SO(N-1)$, with a natural complex structure. Our construction suggests a new class of nonlinear models generalizing the model first proposed in [6]. These models may have interesting properties, including the possibility of solitons with nontrivial topological structures [7].

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